Algebraic Analysis for beginners

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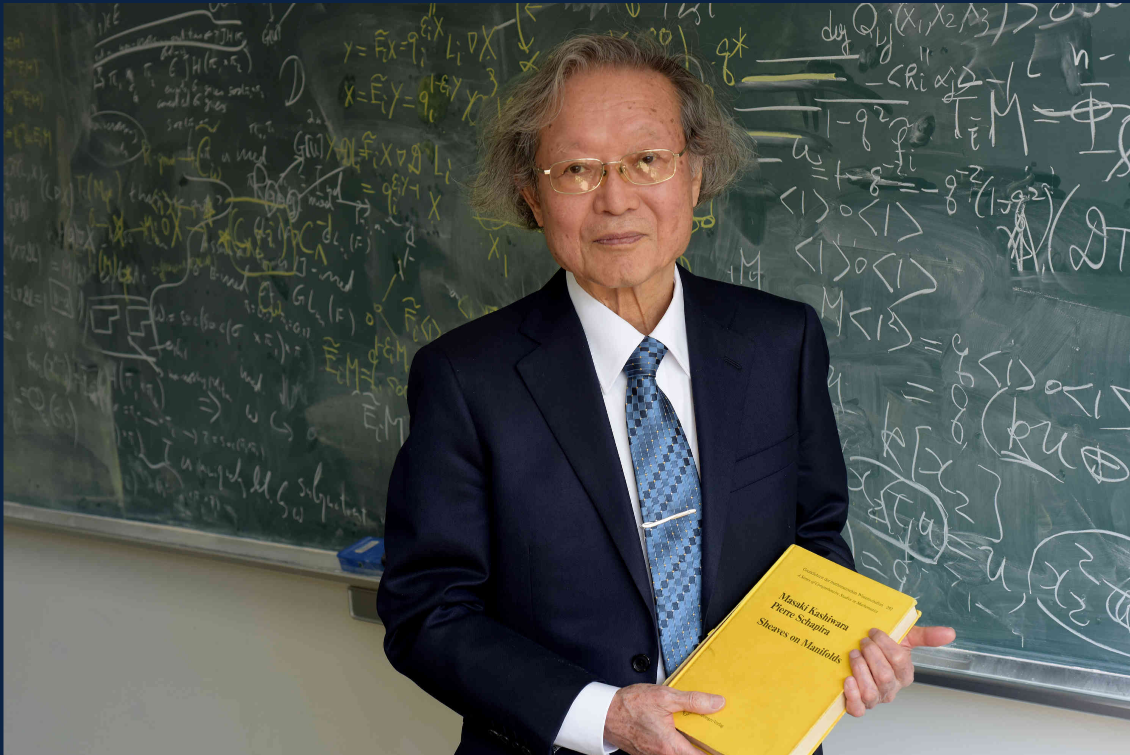
University of Padua

The Abel lectures, Oslo, 21 May 2025

A beautiful picture



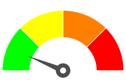
Counting problems

tsurukamezan
鶴亀算 **Counting cranes and turtles:**

Some cranes and turtles dwell around a pond.
There are 18 legs and 7 heads in total.
How many cranes and how many turtles are there?

Children are taught how to find a solution.

Drawing of crane and turtle from `en.ac-illust.com`



math ease gauge

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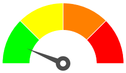

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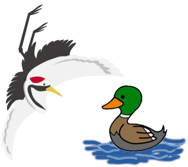
Counting problems



tsurukamozan
鶴鴨算 **Counting cranes and ducks:**

Some cranes and ducks dwell around a pond.
There are 18 legs and 9 heads in total.
How many cranes and how many ducks are there?

Children are taught how to find a solution.

Question: can there be multiple solutions?

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Drawing of crane and turtle from `en.ac-illust.com`
Drawing of duck from `www.emilydrawing.com`

Counting problems

tsurukamezan
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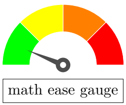

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Counting pairs of socks:

It's your turn to fold the laundry.
There is a total of 18 socks.
How many pairs of socks are there?

Drawing of crane and turtle from `en.ac-illust.com`
Drawing of socks from `juniatapark.philasd.org`

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Counting problems

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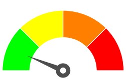
Claim: the **Question** gets the same answer for these two problems.



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Counting problems

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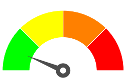
Why? How are the two problems related?



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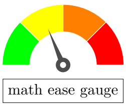
Systems of linear equations

Abelian groups

Children count on their fingers $F = \{1, 2, 3, \ldots\}$.



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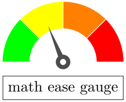
Let's use $\color{red}{F} = \mathbb{Z}$ integers, or another *abelian group* (a set with $+, -$)



Systems of linear equations

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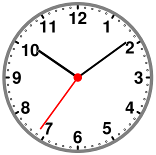
Let's use $\color{red}{F}$ $= \mathbb{Z}$ integers, or another *abelian group* (a set with $+, -$)

like $\mathbb{Z}/\mathbb{Z}(12)$ clock numbers where $[12] = [0]$.



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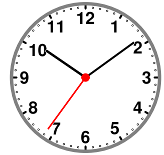
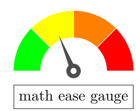


(Q, F)

$\overbrace{s}^{\text{socks}}, \underbrace{p}_{\text{pairs}} \in F: \quad \left\{ 2 \cdot p = s \right. \qquad Q = \begin{pmatrix} 2 \end{pmatrix}$ coefficient.



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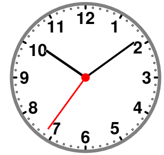

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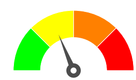

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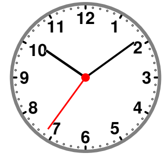

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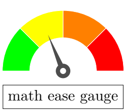
Question: for given ℓ, h in F,
does the linear system (P, F) have multiple solutions?



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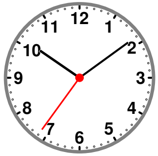

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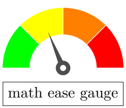

Systems of linear equations (continued)

Abelian groups

F: an abelian group.

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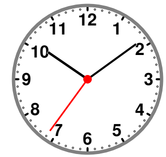

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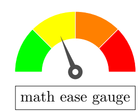
Question: do the systems (P, F) or (Q, F) have multiple solutions?



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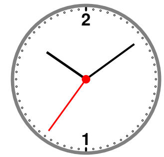

(Q, F)

	$p \in F$: $\begin{cases} 2 \cdot p = 0 \end{cases}$	$Q = (2)$.

$Q \rightsquigarrow M_Q = \mathbb{Z}/\mathbb{Z}\,Q$ "clock numbers" $[m]$ where $\underbrace{[2] = [0]}_{\text{parity check}}$.

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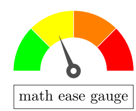

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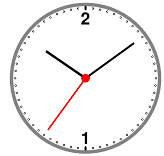

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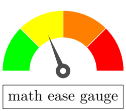

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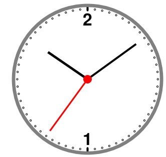

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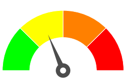

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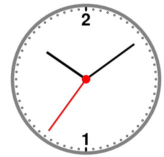

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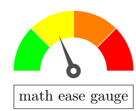
Claim: it depends *only* on (M_P, F) or (M_Q, F), respectively.



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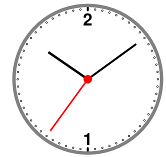

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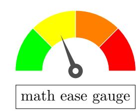
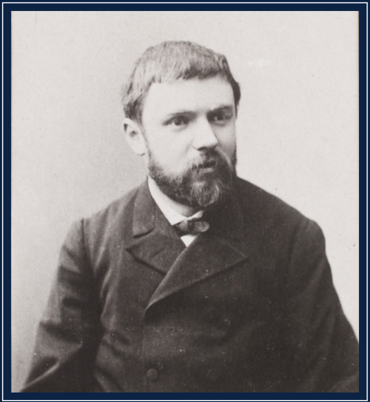
Claim: it depends *only* on (M_P, F) or (M_Q, F), respectively.

Answer: since (Q, F) has a unique solution, so does (P, F).



A categorical remark



Henri Poincaré (1854–1912)

> Les mathématiciens n'étudient pas des objets, mais des relations entre les objets ; il leur est donc indifférent de remplacer ces objets par d'autres, pourvu que les relations ne changent pas.

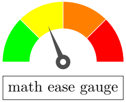
Mathematicians do not study objects, but relations between objects; it is thus indifferent to them to replace these objects by others, as long as the relations do not change.

H. Poincaré, *La Science et l'Hypothèse*, (1902).
Portrait by Eugène Pirou circa 1900, Musée d'Orsay, Paris. Permission requested.

Systems of linear equations

Modules over a ring

$\mathbb{Z}$ is a ring: a set with $+$, $-$, $\overbrace{\cdot}^{\text{product}}$.



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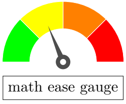
F, M_P are $\mathbb{Z}$-modules: sets with $+, -$, and product by ring elements.



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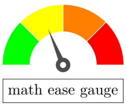
$\mathbb{Z}$-modules form an abelian category: a *set* with $\underbrace{\to}_{\text{morphisms}}$ (used in $M_Q \xrightarrow{\sim} M_P$).



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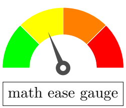
R: a ring, F a $\overbrace{\textit{left}}^{\text{product in } R \text{ may not be commutative}}$ R-module.



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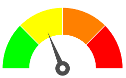
(P, F) $\boxed{u_1, \ldots, u_N \in F: \quad \left\{ \sum_{j=1}^{N} P_{ij}\, u_j = 0 \atop (i=1,\ldots,N') \right. \qquad P = (P_{ij}) \text{ coefficients in } R.}$



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$\mathbb{Z}$ is a ring: a set with $+$, $-$, $\overbrace{\cdot}^{\text{product}}$.

F, M_P are $\mathbb{Z}$-modules: sets with $+, -$, and product by ring elements.

$\mathbb{Z}$-modules form an abelian category: a *set* with $\underbrace{\to}_{\text{morphisms}}$ (used in $M_Q \xrightarrow{\sim} M_P$) .

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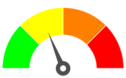

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Systems of linear equations (continued)

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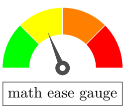

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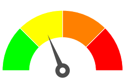

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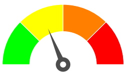
Corollary: $M_P \simeq M_Q \implies \boxed{P\, u = 0}$ and $\boxed{Q\, v = 0}$ have interchangeable solutions.



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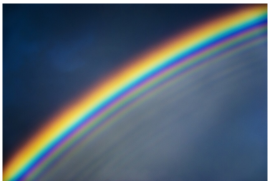
D-modules

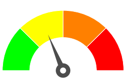
Let us consider linear differential equations, like:



Let $u(\overbrace{x}^{\text{radial distance}})$ be the intensity of colored light in a supernumerary rainbow.

Airy's equation gives: $(\frac{d}{dx})^2\, u = x\, u$.

G.B. Airy, *On the Intensity of Light in the neighbourhood of a Caustic*, (1838).
Photo of rainbows from `commons.wikimedia.org`



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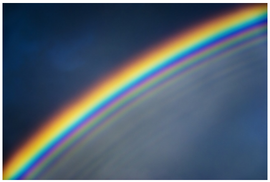

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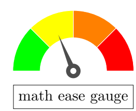

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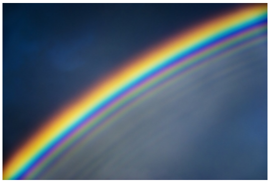

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Leibniz's rule $\quad \frac{d}{dx}(x\, u) = x\, \frac{du}{dx} + u \quad$ implies $\quad \boxed{\partial \cdot x = x \cdot \partial + 1}$

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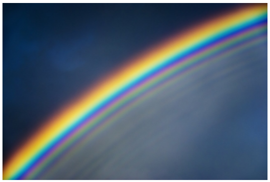

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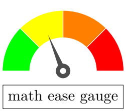

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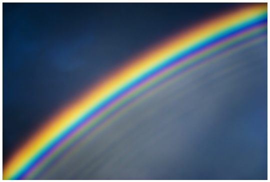

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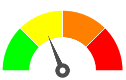
For $P = \partial^2 - x$ in $\mathcal{D}$, Airy's equation is written $\quad \boxed{P\, u = 0}$



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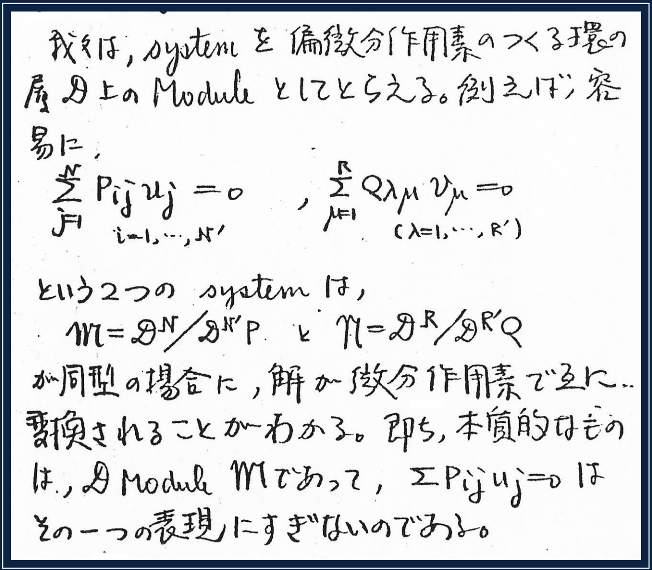
An impressive Master's thesis


我々は，system を偏微分作用素のつくる環の層 $\mathcal{D}$ 上の Module としてとらえる。例えば，容易に，

$$\sum_{j=1}^{N} P_{ij} u_j = 0 \quad (i=1,\cdots,N'), \qquad \sum_{\mu=1}^{R} Q_{\lambda\mu} v_\mu = 0 \quad (\lambda=1,\cdots,R')$$

という２つの system は，

$$\mathfrak{M} = \mathcal{D}^N/\mathcal{D}^{N'}P \quad と \quad \mathfrak{N} = \mathcal{D}^R/\mathcal{D}^{R'}Q$$

が同型の場合に，解が微分作用素で互に変換されることがわかる。即ち，本質的なものは，$\mathcal{D}$ Module $\mathfrak{M}$ であって，$\sum P_{ij} u_j = 0$ はその一つの表現にすぎないのである。

Let $\mathcal{D}$ be the sheaf of rings of partial differential operators.
The systems with coefficients

$$P = (P_{ij}), \quad Q = (Q_{\lambda\mu}),$$

correspond to the modules

$$\underbrace{\mathcal{D}^N/\mathcal{D}^{N'}P}_{\mathcal{M}_P}, \quad \underbrace{\mathcal{D}^R/\mathcal{D}^{R'}Q}_{\mathcal{M}_Q}.$$

If the modules are isomorphic, the systems' solutions are interchanged using differential operators.
What is essential is the $\mathcal{D}$-module $\mathcal{M}_P$, not its presentation $\sum P_{ij} u_j = 0$.

Masaki Kashiwara, *Algebraic study of systems of partial differential equations*, Master's thesis, (1970)

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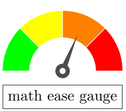

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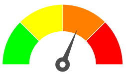

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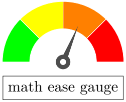

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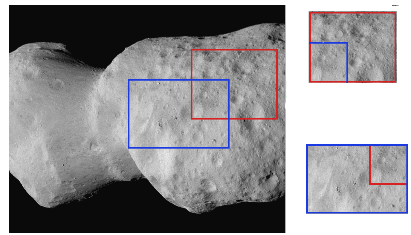

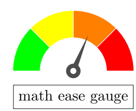

Picture of asteroid Donaldjohanson on 20 April 2025 from `science.nasa.gov`

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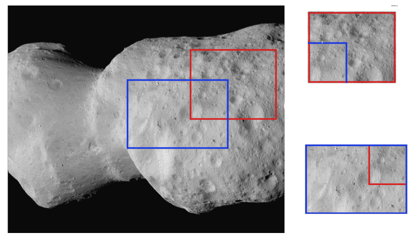

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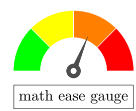

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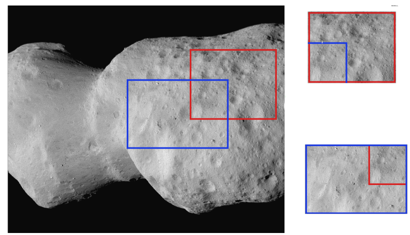

$\mathcal{O}_X$: sheaf of holomorphic functions on X.

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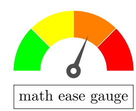
i.e., locally, *polynomials* in $a(x)$, $\partial_i = \frac{\partial}{\partial x_i}$ $(i = 1, \ldots, n)$.



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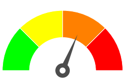
A microlocal perspective: Characteristic variety

$X \ni x, \quad T^*X \ni (\overbrace{x}^{\text{position}}\,;\,\overbrace{\xi}^{\text{momentum}})$ cotangent bundle (phase space).



math ease gauge

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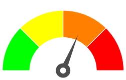

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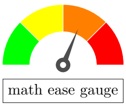

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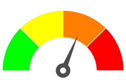

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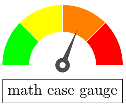

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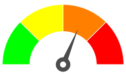

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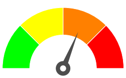

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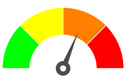

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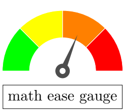

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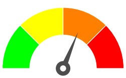

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> We received the final manuscript in June, 1971 but have postponed the publication because the authors had the intention of adding an introduction to the paper. Since we do not think it appropriate to wait for it forever, we have decided to publish this part in the present form.
>
> December 28, 1972
>
> Hikosaburo Komatsu

From the introduction to the Lecture Notes.

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a.k.a. **Hilbert's 21st problem**

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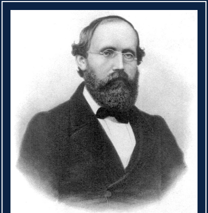

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(1826–1866)

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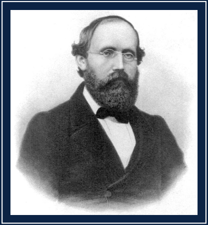

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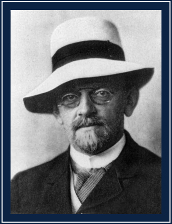

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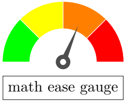

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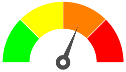

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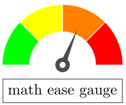

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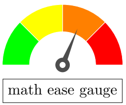

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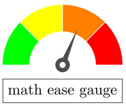

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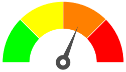

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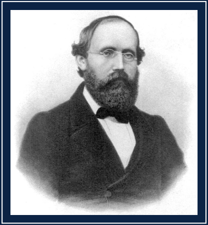

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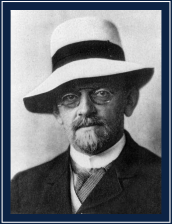

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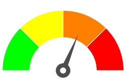

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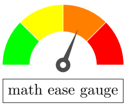

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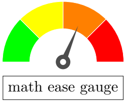

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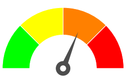

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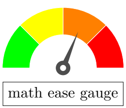

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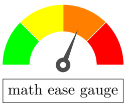

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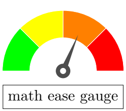

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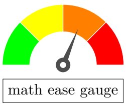

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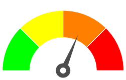

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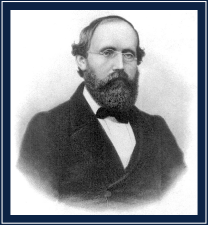

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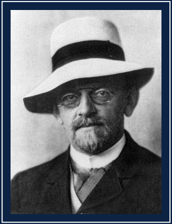

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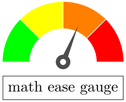

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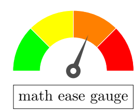

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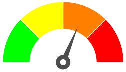

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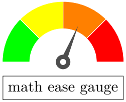

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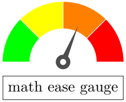

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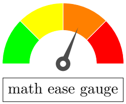

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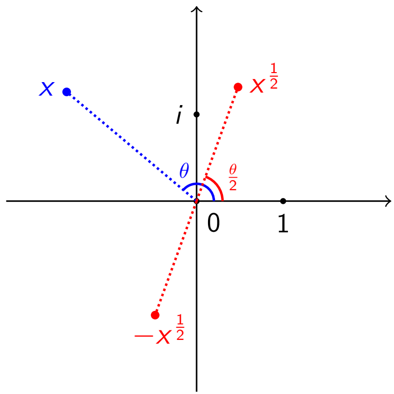

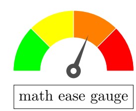

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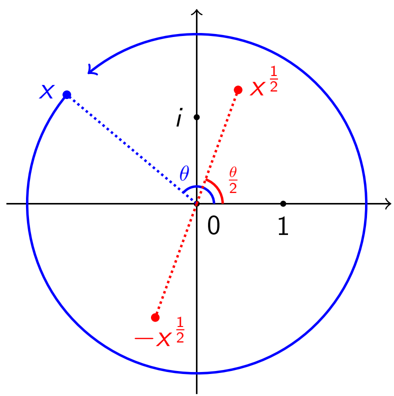

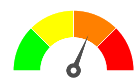
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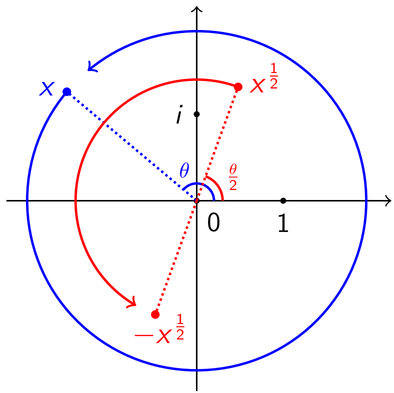

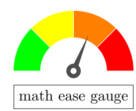

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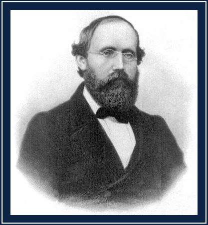

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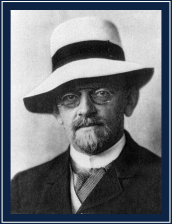

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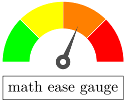

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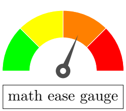

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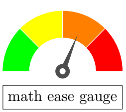

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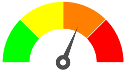
Fact: $\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_P, \mathcal{O}_X) \simeq \mathcal{H}om_{\mathcal{D}}(\mathcal{M}_Q, \mathcal{O}_X)$ as $\mathbb{C}_X$-modules.



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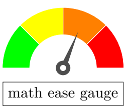


A variation on $\mathcal{M}_P \simeq \mathcal{M}_Q$

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2aD_x K_0^*(2\sqrt{aD_x}) & 2\sqrt{aD_x}K_1^*(2\sqrt{aD_x}) \\ -I_0(2\sqrt{aD_x}) & \frac{1}{\sqrt{aD_x}} I_1(2\sqrt{aD_x}) \end{pmatrix} \begin{pmatrix} u \\ -xD_x u \end{pmatrix}.$$

Here $\quad I_\nu(\tau) \equiv \left(\frac{\tau}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(\tau/2)^{2n}}{n!\Gamma(\nu+n+1)}$ and

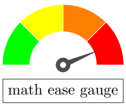
$$\begin{aligned} K_n^*(\tau) &\equiv (-1)^n I_n(\tau) \log(\tau/2) + K_n(\tau) \\ &= \frac{(-1)^n}{2} \sum_{k=0}^{\infty} \frac{\psi(k+1)+\psi(k+n+1)}{k!(n+k)!} \left(\frac{\tau}{2}\right)^{n+2k} \\ &\qquad + \frac{1}{2} \sum_{r=0}^{n-1} (-1)^r \frac{(n-r-1)!}{r!} \left(\frac{\tau}{2}\right)^{2r-n}, \end{aligned}$$

where $\psi(n) = \sum_{k=0}^{n-1} \frac{1}{k} - \gamma$ with Euler's constant $\gamma = 0.57721\cdots$.

M. Kashiwara and T. Kawai, *On Holonomic Systems of Microdifferential Equations III*, (1981)

Perverse sheaves

$$X = \mathbb{C}, \quad P_\lambda = x\partial - \lambda, \quad \mathrm{char}(\mathcal{M}_{P_\lambda}) = \overbrace{\{x\xi = 0\}}^{\text{singularity } \{x=0\} \text{ included}},$$



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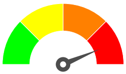
$$L_{P_\lambda} = \mathcal{H}om_{\mathcal{D}}(\mathcal{M}_{P_\lambda}, \mathcal{O}_X) = \ker\big(\, \mathcal{O}_X \xrightarrow{P_\lambda \cdot} \mathcal{O}_X \,\big) = \{u_\lambda \in \mathcal{O}_X : P_\lambda\, u_\lambda = 0\},$$



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Perverse sheaves

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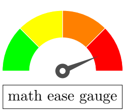

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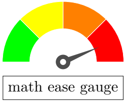
Exceptions: $P_{-1} = x\partial + 1$ | $P_0 = x\partial$



Perverse sheaves

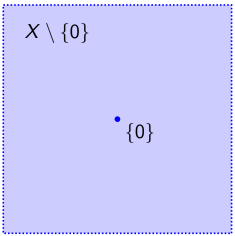
$$X = \mathbb{C}, \quad {\color{red}P_\lambda} = x\partial - \lambda, \quad \operatorname{char}({\color{red}\mathcal{M}_{P_\lambda}}) = \overbrace{\{x\xi = 0\}}^{\text{singularity } \{x=0\} \text{ included}},$$

$$ {\color{red}L_{P_\lambda}} = \mathcal{H}om_{\mathcal{D}}(\mathcal{M}_{P_\lambda}, \mathcal{O}_X) = \ker\big(\,\mathcal{O}_X \xrightarrow{P_\lambda \cdot} \mathcal{O}_X\,\big) = \{u_\lambda \in \mathcal{O}_X : P_\lambda\, u_\lambda = 0\},$$

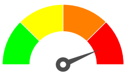
Fact: for ${\color{red}\lambda \neq -1}$, $\mathcal{M}_{P_\lambda} \simeq \mathcal{M}_{P_{\lambda+1}}$: $\quad \underbrace{u_\lambda}_{\text{locally: } c\,x^\lambda} \mapsto x \cdot u_\lambda, \quad \underbrace{u_{\lambda+1}}_{\text{locally: } c\,x^{\lambda+1}} \mapsto \frac{1}{\lambda+1}\,\partial \cdot u_{\lambda+1}.$

Exceptions: $P_{-1} = x\partial + 1$ | $P_0 = x\partial$

$L_{P_{-1}} = {\color{blue}\mathbb{C}_{X \setminus \{0\}}}$ | $L_{P_0} = {\color{blue}\mathbb{C}_X}$



stratification of X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Perverse sheaves

$$X = \mathbb{C}, \qquad P_\lambda = x\partial - \lambda, \qquad \mathrm{char}(\mathcal{M}_{P_\lambda}) = \overbrace{\{x\xi = 0\}}^{\text{singularity } \{x=0\} \text{ included}},$$

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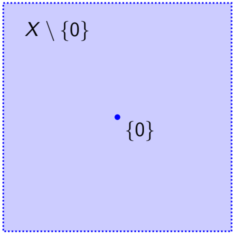
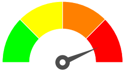
Exceptions: $P_{-1} = x\partial + 1$ | $P_0 = x\partial \qquad (\mathcal{M}_\partial \simeq \mathcal{O}_X)$

$L_{P_{-1}} = \mathbb{C}_{X\setminus\{0\}}$ | $L_{P_0} = \mathbb{C}_X = L_\partial$



stratification of X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Perverse sheaves

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$${\color{red}H^0 L_{P_\lambda}} = \mathcal{H}om_{\mathcal{D}}(\mathcal{M}_{P_\lambda}, \mathcal{O}_X) = \ker\left(\mathcal{O}_X \xrightarrow{P_\lambda \cdot} \mathcal{O}_X \right) = \{u_\lambda \in \mathcal{O}_X : P_\lambda u_\lambda = 0\},$$

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Exceptions:

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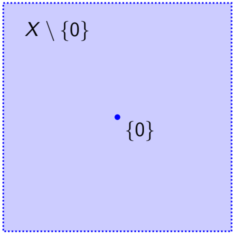
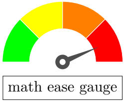
$H^0 L_{P_{-1}} = {\color{blue}\mathbb{C}_{X \setminus \{0\}}}$	$H^0 L_{P_0} = {\color{blue}\mathbb{C}_X} {\color{red}\, = H^0 L_\partial}$



stratification of X



Perverse sheaves

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$${\color{red}H^0L_{P_\lambda}}=\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_{P_\lambda},\mathcal{O}_X)=\ker\left(\mathcal{O}_X\xrightarrow{P_\lambda\cdot}\mathcal{O}_X\right)=\{u_\lambda\in\mathcal{O}_X:\ P_\lambda u_\lambda=0\},$$

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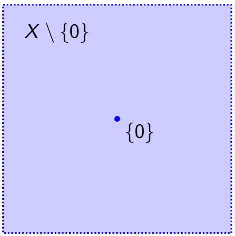
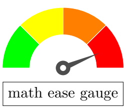
$P_{-1}=x\partial+1$	$P_0=x\partial\qquad {\color{red}(\mathcal{M}_\partial\simeq\mathcal{O}_X)}$
$H^0L_{P_{-1}}={\color{blue}\mathbb{C}_{X\setminus\{0\}}}$	$H^0L_{P_0}={\color{blue}\mathbb{C}_X}\ {\color{red}=H^0L_\partial}$



stratification of X



Perverse sheaves

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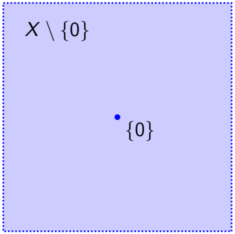
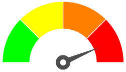
$P_{-1} = x\partial + 1$	$P_0 = x\partial \qquad \color{red}{(\mathcal{M}_\partial \simeq \mathcal{O}_X)}$
$H^0 L_{P_{-1}} = \mathbb{C}_{X \setminus \{0\}}$	$H^0 L_{P_0} = \mathbb{C}_X \color{red}{= H^0 L_\partial}$
$H^1 L_{P_{-1}} = 0$	$H^1 L_{P_0} = \mathbb{C}_{\{0\}}$



stratification of X


math ease gauge

Perverse sheaves

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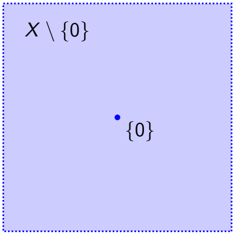

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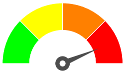

math ease gauge

Perverse sheaves

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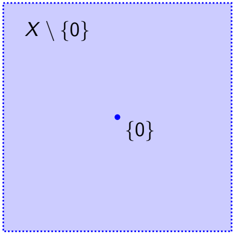
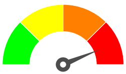


stratification of X

$$L_{P_\lambda} = R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_{P_\lambda}, \mathcal{O}_X) = \big(\,\mathcal{O}_X \xrightarrow{P_\lambda\cdot} \mathcal{O}_X\,\big): \text{ a perverse sheaf}.$$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Holonomic $\mathcal{D}$-modules

Theorem: $\mathcal{M}$ coherent $\implies$ $\mathrm{char}(\mathcal{M}) \subset T^*X$ is involutive.

In particular, $\boxed{\dim \mathrm{char}(\mathcal{M}) \geq \dim X} = \frac{1}{2} \dim T^*X$.

M. Sato, T. Kawai and M. Kashiwara, *Microfunctions and pseudo-differential equations*, (1973)

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In particular, $\boxed{\dim \mathrm{char}(\mathcal{M}) \geq \dim X} = \frac{1}{2} \dim T^*X$.

$\mathcal{M}$ $\underbrace{\text{holonomic}}_{\text{maximally overdetermined}}$: coherent & $\dim \underbrace{\mathrm{char}(\mathcal{M})}_{\text{Lagrangian}} = \dim X$.

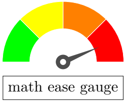
M. Sato, T. Kawai and M. Kashiwara, *Microfunctions and pseudo-differential equations*, (1973)

Riemann-Hilbert correspondence

$$\overbrace{\textbf{Riemann-Hilbert:}}^{\text{non singular}} \quad \{\text{non singular } \mathcal{D}_X\text{-modules}\} \underset{\sim}{\rightleftarrows} \{\text{local systems}\}^{\mathrm{op}}$$

$$\mathcal{M} \longmapsto \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_X)$$

$$\mathcal{H}om_{\mathbb{C}}(L, \mathcal{O}_X) \longleftarrow\!\shortmid L$$



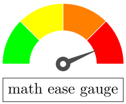
Riemann-Hilbert correspondence

$$\overbrace{\textbf{Riemann-Hilbert:}}^{\text{regular}}\ \{\text{regular holonomic } \mathcal{D}_X\text{-modules}\} \underset{\sim}{\rightleftarrows} \{\text{perverse sheaves}\}^{\mathrm{op}}$$

$$\mathcal{M} \longmapsto R\,\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_X)$$

$$R\,\mathcal{H}om_{\mathbb{C}}(L, \mathcal{O}_X^t) \longleftarrow\!\shortmid\ L$$

M. Kashiwara, *The Riemann-Hilbert problem for holonomic systems,* (1984).



Riemann-Hilbert correspondence

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$$R\mathcal{H}om_{\mathbb{C}}(L, \mathcal{O}_X^t) \longleftarrow\!\shortmid L$$

$\mathcal{O}_X^t$: ind-sheaf of "tempered holomorphic functions"

M. Kashiwara, *The Riemann-Hilbert problem for holonomic systems*, (1984).
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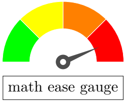

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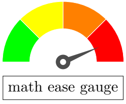
$\mathcal{O}_X^t$: ind-sheaf of “tempered holomorphic functions”

Example: $X = \mathbb{C}, \quad P_{-1} = x\partial + 1, \quad \mathcal{M}_{P_{-1}} \mapsto \mathbb{C}_{X\setminus\{0\}}$:



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Riemann-Hilbert correspondence

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$$\mathcal{M} \longmapsto R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_X)$$

$$R\mathcal{H}om_{\mathbb{C}}(L, \mathcal{O}_X^t) \longleftarrow\!\shortmid\ L$$

$\mathcal{O}_X^t$: ind-sheaf of “tempered holomorphic functions”

Example: $X = \mathbb{C}, \quad P_{-1} = x\partial + 1, \quad \mathcal{M}_{P_{-1}} \mapsto \mathbb{C}_{X\setminus\{0\}}$:

$$R\mathcal{H}om_{\mathbb{C}}(\mathbb{C}_{X\setminus\{0\}}, \mathcal{O}_X) \ni \exp(1/x) \quad \text{essential singularities at } \{0\},$$

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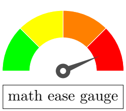

Riemann-Hilbert correspondence

$$\overbrace{\textbf{Riemann-Hilbert:}}^{\text{regular}} \ \{\text{regular holonomic } \mathcal{D}_X\text{-modules}\} \underset{\sim}{\rightleftarrows} \{\text{perverse sheaves}\}^{\mathrm{op}}$$

$$\mathcal{M} \longmapsto R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_X)$$

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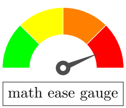
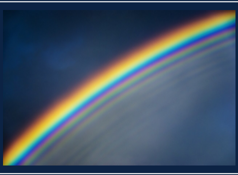


Stokes phenomenon



Let u be the intensity of colored light in a supernumerary rainbow.
It solves Airy's equation $P\,u = 0$, with $P = \partial^2 - x$ irregular at $x = \infty$.

G.B. Airy, *On the Intensity of Light in the neighbourhood of a Caustic*, (1838).

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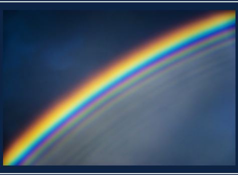

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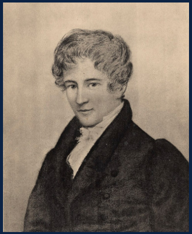

Niels Abel (1802–1829)

"Divergent series are, in general, something terrible and it is a shame to base any proof on them… For the most part, it is true that the results are correct, which is very strange. I am working to find out why, a very interesting problem."

Letter from Abel to Holmboe (1826)

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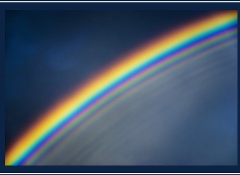

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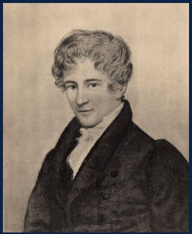

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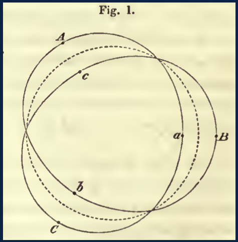
Letter from Abel to Holmboe (1826)

George Stokes (1819–1903)

". . . the inferior term enters as it were into a mist, is hidden for a little from view, and comes out with its coefficient changed."

Comments on his 1857 paper from a volume of Acta Mathematica dedicated to Abel (1902)



G.B. Airy, *On the Intensity of Light in the neighbourhood of a Caustic*, (1838).
G.G. Stokes, *On the discontinuity of arbitrary constants which appear in divergent developments*, (1857).

Riemann-Hilbert correspondence

$\overbrace{\textbf{Riemann-Hilbert:}}^{\text{regular}}$ $\{\text{regular holonomic } \mathcal{D}_X\text{-modules}\} \underset{\longleftarrow}{\xrightarrow{\sim}} \{\text{perverse sheaves}\}^{\mathrm{op}}$

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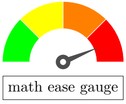

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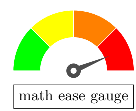
$\overbrace{\textbf{Riemann-Hilbert:}}^{\text{irregular}}$ $\{\text{holonomic } \mathcal{D}_X\text{-modules}\} \underset{\sim}{\rightleftarrows} \{\text{enhanced perverse ind-sheaves}\}^{\mathrm{op}}$.



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A. D'Agnolo and M. Kashiwara, *Riemann-Hilbert correspondence for holonomic D-modules,* (2016).

A titillating title

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Bravo Masaki for your Abel Prize!